Assignment 3

Hand in no. 5, 7 and 8 by Sept 26, 2019.

1. A function on $[a,b]$ is called Hölder continuous at $x \in [a,b]$ if there are $\alpha \in (0,1), L$ and δ such that $|f(y)-f(x)| \le L|y-x|^\alpha$ for all $y \in [a,b], |y-x| < \delta$. Prove that Theorem 1.5 holds when "Lipschitz continuous" is replaced by "Hölder continuous".

2. Let f be a function defined on (a,b) and $x_0 \in (a,b)$.

   (a) Show that f is Lipschitz continuous at x_0 if its left and right derivatives exist at x_0.

   (b) Construct a function Lipschitz continuous at x_0 whose one sided derivatives do not exist.

3. Let f be a function defined on $(a,b]$ which is integrable on $[c,b]$ for all $c \in (a,b)$. It is called improperly integrable over $(a,b]$ if

$$\lim_{c \to a^+} \int_c^b |f|$$

   exists. When this happens,

$$\lim_{c \to a^+} \int_c^b f$$

   also exists and we define the improper integral of f over $(a,b]$ to be

$$\int_a^b f = \lim_{c \to a^+} \int_c^b f \ .$$

   (a) Show that if f is integrable on $[a,b]$, its improper integral also exists and is equal to it usual integral.

   (b) Show that Riemann-Lebesgue Lemma holds for improperly integrable functions.

4. Optional. Show that

$$-\log \left| 2 \sin \frac{x}{2} \right| \sim \cos x + \frac{\cos 2x}{2} + \frac{\cos 3x}{3} + \cdots \ .$$

   Suggestion. Verify this function is 2π-periodic and improperly integrable first. The calculation of a_0 is tricky, involving the definite integral $I = \int_0^{\pi/2} \log \sin t dt$. To evaluate it use $\sin t = 2 \sin t/2 \cos t/2$ and eventually show $I = -\frac{\pi}{2} \log 2$.

5. Let a_n, b_n be the Fourier coefficients of some $f \in R_{2\pi}$.

   (a) Show that for each $r \in [0,1)$, the trigonometric series given by

$$a_0 + \sum_{k=1}^{\infty} r^n (a_n \cos nx + b_n \sin nx)$$

   is uniformly convergent to some function in $C_{2\pi}$. Denote this function by $f_r(x)$.

(b) Show that

$$f_r(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(z)f(x+z)dz,$$

where the **Poisson kernel** P_r is given by

$$P_r(z) = \frac{1-r^2}{1-2r\cos z + r^2} \ .$$

(c) Let f be continuous at x. Show that $\lim_{r\to 1} f_r(x) = f(x)$.

The treatment is parallel to that for the Dirichlet kernel (the parameter n is now replaced by r), but differs at the final step; we do not need Lipschitz continuity. Think about it.

6. (a) Can you find a cosine series which converges uniformly to the sine function on $[0,\pi]$? If yes, find one.
   (b) Is the series in (a) unique?
   (c) Can you find a cosine series which converges pointwisely to the sine function on $[-a,\pi]$ where a is a number in $(0,\pi)$?

7. Let f be an integrable function on $[-\pi,\pi]$. Show that for each $\varepsilon > 0$, there exists a trigonometric polynomial p satisfying $p < f$ on $[-\pi,\pi]$ and

$$\int_{-\pi}^{\pi} |f-p| < \varepsilon \ .$$

8. Show that there is a countable subset of $C[a,b]$ such that for each $f \in C[a,b]$, there is some $\varepsilon > 0$ such that $\|f-g\|_\infty < \varepsilon$ for some g in this set. Suggestion: Take this set to be the collection of all polynomials whose coefficients are rational numbers.

9. Optional. Let f be continuously on $[a,b]\times[c,d]$. Show that for each $\varepsilon > 0$, there exists a polynomial $p = p(x,y)$ so that

$$\big\|f-p\big\|_\infty < \varepsilon, \quad \text{in } [a,b]\times[c,d].$$

In fact, this result holds in arbitrary dimension.